Select Solutions (Curl and Divergence)

(1d) Compute curl and divergence of $\vec{F} = \langle \ln(2y+3z), \ln(x+3z), \ln(x+2y) \rangle$.

Sol: $\text{div}(\vec{F}) = \vec{\nabla} \cdot \vec{F}$

$$= \frac{\partial}{\partial x}\left[\ln(2y+3z)\right] + \frac{\partial}{\partial y}\left[\ln(x+3z)\right] + \frac{\partial}{\partial z}\left[\ln(x+2y)\right]$$

$$= 0 + 0 + 0$$

$$= 0$$

$$\text{Curl}(\vec{F}) = \vec{\nabla} \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \ln(2y+3z) & \ln(x+3z) & \ln(x+2y) \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}\left[\ln(x+2y)\right] - \frac{\partial}{\partial z}\left[\ln(x+3z)\right], -\left(\frac{\partial}{\partial x}\left[\ln(x+2y)\right] - \frac{\partial}{\partial z}\left[\ln(2y+3z)\right]\right), \frac{\partial}{\partial x}\left[\ln(x+3z)\right] - \frac{\partial}{\partial y}\left[\ln(2y+3z)\right] \right\rangle$$

$$= \left\langle \frac{2}{x+2y} - \frac{3}{x+3z}, -\left(\frac{1}{x+2y} - \frac{3}{2y+3z}\right), \frac{1}{x+3z} - \frac{2}{2y+3z} \right\rangle$$

$$= (x+2y)^{-1}(x+3z)^{-1}(2y+3z)^{-1}\left\langle 2(x+3z) - 3(x+2y), -\left((2y+3z) - 3(x+2y)\right), (2y+3z) - 2(x+3z) \right\rangle$$

$$= (x+2y)^{-1}(x+3z)^{-1}(2y+3z)^{-1}\left\langle -x-6y+6z, 3x+4y-3z, -2x+2y-3z \right\rangle$$ ▨

(2f) Determine whether $\vec{F} = e^x \langle \sin(yz), z\cos(yz), y\cos(yz) \rangle$ is conservative or not. If conservative, compute its potential.

<u>Sol</u>: We first compute $\text{curl}(\vec{F})$ to check conservativity:

$$\text{Curl}(\vec{F}) = \vec{\nabla} \times \vec{F}$$

$$= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin(yz) & ze^x \cos(yz) & ye^x \cos(yz) \end{bmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}\left[ye^x \cos(yz)\right] - \frac{\partial}{\partial z}\left[ze^x \cos(yz)\right], -\left(\frac{\partial}{\partial x}\left[ye^x \cos(yz)\right] - \frac{\partial}{\partial z}\left[e^x \sin(yz)\right]\right), \frac{\partial}{\partial x}\left[ze^x \cos(yz)\right] - \frac{\partial}{\partial y}\left[e^x \sin(yz)\right] \right\rangle$$

$$= \left\langle \left(e^x \cos(yz) - yze^x \sin(yz)\right) - \left(e^x \cos(yz) - yze^x \sin(yz)\right), \right.$$
$$\left. \left(ye^x \cos(yz)\right) - \left(ye^x \cos(yz)\right), \quad \left(ze^x \cos(yz)\right) - \left(ze^x \cos(yz)\right) \right\rangle$$

$$= \langle 0, 0, 0 \rangle = \vec{0} \qquad \therefore \ \vec{F} = \nabla f \text{ is conservative.}$$

Now $\frac{\partial f}{\partial x} = e^x \sin(yz)$ implies $f(x,y,z) = \int e^x \sin(yz)\, dx = e^x \sin(yz) + C(y,z)$.

$\therefore \ ze^x \cos(yz) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[e^x \sin(yz) + C(y,z)\right] = ze^x \cos(yz) + C_y(y,z)$.

So $C_y(y,z) = 0$, yielding $C(y,z) = \int 0\, dy = D(z)$.

$\therefore \ f(x,y,z) = e^x \sin(yz) + D(z)$. Moreover

$ye^x \cos(yz) = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[e^x \sin(yz) + D(z)\right] = ye^x \cos(yz) + D'(z)$,

So $D'(z) = 0$ yields $D(z) = \int 0\, dz = E$.

Hence we have potential function $f(x,y,z) = e^x \sin(yz)$. ☐